On mutual information

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Abstract

1 mutual information versus mixing

Let (Ω, μ, T) be a ϕ-mixing dynamical system, where μ is an invariant measure. Let A be a measurable partitioning of Ω and let $A^n = A \vee T^{-1}A \vee T^{-2}A \vee \cdots \vee T^{-n}A$. Let $H(A^n, A^{-\infty})$ be the mutual information between the past $A^{-\infty}$ and the future A^n. We want to show that $\lim_{n\to\infty} \frac{H(A^n, A^{-\infty})}{n} = 0$.

We begin the proof with the following string of equalities/inequalities:

$$
\begin{aligned}
H(A^n, A^{-\infty}) &= H(A^n) - H(A^n|A^{-\infty}) \\
&= H(A^\Delta \vee T^{-\Delta}A^{n-\Delta}) - H(A^n|A^{-\infty}) \\
&\leq H(T^{-\Delta}A^{n-\Delta}) + H(A^\Delta|T^{-\Delta}A^{n-\Delta}) - H(A^n|A^{-\infty}) \\
&\leq H(A^{n-\Delta}) + H(A^\Delta) - H(A^n|A^{-\infty}) \\
&= H(A^{n-\Delta}) + H(A^\Delta) - H(A^\Delta \vee T^{-\Delta}A^{n-\Delta} \\
&\leq H(A^{n-\Delta}) + H(A^\Delta) - H(T^{-\Delta}A^{n-\Delta})
\end{aligned}
$$

It follows that

$$\frac{H(A^n, A^{-\infty})}{n} = \frac{H(A^{n-\Delta}) - H(T^{-\Delta}A^{n-\Delta})}{n} + \frac{H(A^\Delta)}{n}$$

Since $\frac{H(A^\Delta)}{n} \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \frac{H(A^n, A^{-\infty})}{n} = \lim_{n\to\infty} \frac{H(A^{n-\Delta}) - H(T^{-\Delta}A^{n-\Delta})}{n}$$

The numerator of the right hand side of the latter is now evaluated:

$$
\begin{aligned}
&H(T^{-\Delta}A^{n-\Delta}|A^{-\infty}) \\
&= \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\frac{\mu(A\cap B)}{\mu(B)} \\
&= \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}\mu(A)\right) \\
&= \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\mu(A) + \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\left(1+\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)\right) \\
&= H(T^{-\Delta}A^{n-\Delta}) + \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\left(1+\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)\right) \\
&= H(A^{n-\Delta}) + \sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\left(1+\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)\right)
\end{aligned}
$$

It follows that

$$
\lim_{n\to\infty}\frac{H(A^n, A^{-\infty})}{n} = -\lim_{n\to\infty}\frac{1}{n}\sum_{A\in T^{-\Delta}A^{n-\Delta}, B\in A^{-\infty}} \mu(A\cap B)\log\left(1+\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)\right)
$$

The expression $\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)$ is now evaluated using mixing. A typical element $A\in T^{-\Delta}A^{n-\Delta}$ is of the form $T^{-\Delta}C$ for some $C\in A^{n-\Delta}$. Hence,

$$
\begin{aligned}
&\log\left(1+\left(\frac{\mu(A\cap B)}{\mu(A)\mu(B)}-1\right)\right) \\
&= \log\left(1+\left(\frac{\mu(A\cap T^{-\Delta}C)}{\mu(A)\mu(T^{-\Delta}C)}-1\right)\right) \\
&= \log\left(1+\left(\frac{\mu(A\cap T^{-\Delta}C)}{\mu(A)\mu(C)}-1\right)\right) \\
&\leq \log(1+\phi(\Delta))
\end{aligned}
$$

Hence

$$
0 \leq \lim_{n\to\infty}\frac{H(A^n, A^{-\infty})}{n} \leq \lim_{n\to\infty}\frac{\log(1+\phi(\Delta))}{n} = 0
$$

and the claim is proved.